Integrals

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In the given interval a function $F(x)$ is called an antiderivative or primitive for the function $f(x)$ if in this interval $f(x)$ is a derivative of a function $F(x)$ that is, if F is differentiable and $F' = f$.

Theorem 1
Two primitive functions $f\colon \mathbb{R} \supseteq D \longrightarrow \mathbb{R}$ differs only by a constant (called a constant of integration). In other words if F and G are primitive functions for a function f, then $F - G = c$ for some $c \in \mathbb{R}$.
Dowód:
Let F and G be primitive functions for the function f. Then

$$(F - G)' = F' - G' = f - f = 0.$$

The derivative of $F - G$ is equal to 0, so it must be constant. Therefore there exists a constant $c \in \mathbb{R}$ such, that $F - G = c$.

□

Corollary:
It is enough to find only one primitive function F for the function f. All other functions will differ only by a constant factor.

An expression $F(x) + c$, where c is a constant is called *indefinite integral* of the function $f(x)$ and is denoted by

$$\int f(x)\,dx.$$

The product $f(x)\,dx$ is called an *integrand expression* and a function $f(x)$ – *an integrand.*

Remarks:

- Integration is a reverse operation to differentiation.
- A primitive function not always exists.
- Every continuous function in the interval $\langle a, b\rangle$ is integrable at this interval (without a proof).

Example 1
Let $f(x) = x$. Then

$$\int f(x)\,dx = \int x\,dx = \frac{1}{2}x^2 + c.$$

Verification

$$\left(\frac{1}{2}x^2 + c\right)' = \frac{1}{2}2x + 0 = x.$$

Example 2
Let $f(x) = x^2$. *Then*

$$\int f(x)\,dx = \int x^2\,dx = \frac{1}{3}x^3 + c.$$

Verification

$$\left(\frac{1}{3}x^3 + c\right)' = \frac{1}{3}3x^2 + 0 = x^2.$$

TABLE OF INTEGRALS OF BASIC FUNCTIONS

$\int 0\,dx = c$	$\int 1\,dx = \int dx = x + c$
$\int x^k\,dx = \frac{1}{k+1}x^{k+1} + c \quad (k \neq -1)$	$\int \frac{1}{x}\,dx = \ln\|x\| + c$
$\int e^x\,dx = e^x + c$	$\int a^x\,dx = \frac{a^x}{\ln a} + c \quad (a > 0,\ a \neq 1)$
$\int \sin x\,dx = -\cos x + c$	$\int \cos x\,dx = \sin x + c$
$\int \tan x\,dx = -\ln\|\cos x\| + c$	$\int \tan^{-1} x\,dx = \ln\|\sin x\| + c$
$\int \frac{1}{\cos^2 x}\,dx = \tan x + c$	$\int \frac{1}{\sin^2 x}\,dx = -\tan^{-1} x + c$
$\int \frac{1}{1+x^2}\,dx = \arctan x + c$	$\int \frac{-1}{1+x^2}\,dx = \arctan x^{-1} + c$
$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c$	$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos x + c$

Integration rules:

I A constant could be taken outside an integral that is, if $a \neq 0$ is a constant, then

$$\int af(x)\,dx = a\int f(x)\,dx$$

II An indefinite integral of a sum (difference) of functions is equal to a sum (difference) of indefinite integrals of those functions taken separately, that is

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Example 3

$$\int (4x^3 - 10x^2 - 5x + 8)\,dx = 4\int x^3\,dx - 10\int x^2\,dx - 5\int x\,dx + 8\int dx = x^4 - \frac{10}{3}x^3 - \frac{5}{2}x^2 + 8x + c.$$

Example 4

$$\int \frac{3x^2 - 7x + 6}{x}dx = \int \left(3x - 7 + \frac{6}{x}\right)dx = \frac{3}{2}x^2 - 7x + 6\ln|x| + c.$$

III *Integration by parts*

Theorem 2
Let f, g be differentiable functions on the given interval D and let there exists an indefinite integral of a function $f \cdot g'$ in this interval. Then there also exists an indefinite integral of a function $f' \cdot g$ on the interval D and

$$\int f'g\,dx = fg - \int fg'\,dx.$$

Proof:
For differentiable functions f, g the formula:

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

is valid. It is equivalent to

$$f' \cdot g = (f \cdot g)' - f \cdot g'.$$

Functions at the right side are integrable, so it is also the function on the left side. What's more

$$\int f' \cdot g\,dx = \int \left[(f \cdot g)' - f \cdot g'\right]\,dx = f \cdot g - \int f \cdot g'\,dx.$$

□

Example 5
Let $f'(x) = \cos x$ and $g(x) = x$. Then $f(x) = \sin x + c$, $g'(x) = 1$ and

$$\int x \cos x\,dx = x \sin x - \int \sin x\,dx = x \sin x + \cos x + c.$$

Example 6
Let $f'(x) = \sin x$ and $g(x) = x^2$. Then $f(x) = -\cos x + c$, $g'(x) = 2x$ and

$$\begin{aligned}\int x^2 \sin x\,dx &= -x^2 \cos x + \int 2x \cos x\,dx = -x^2 \cos x + 2 \int x \cos x\,dx \\ &= -x^2 \cos x + 2\,(x \sin x + \cos x) + c.\end{aligned}$$

Example 7
Let $f'(x) = 1$ and $g(x) = \ln x$. Then $f(x) = x + c$, $g'(x) = 1/x$ and

$$\int \ln x\,dx = \int 1 \cdot \ln x\,dx = x \ln x - \int x \cdot \frac{1}{x}\,dx = x \ln x - \int dx = x \ln x - x + c.$$

Example 8

$$\int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx = e^x \cos x + e^x \sin x - \int e^x \cos x\,dx,$$

therefore

$$2 \int e^x \cos x\,dx = e^x \cos x + e^x \sin x + c,$$

and

$$\int e^x \cos x\,dx = \frac{1}{2} e^x\,(\sin x + \cos x) + c.$$

IV *Integration by substitution*

Theorem 3
Let $I, J \subseteq \mathbb{R}$ be intervals, $G\colon J \longrightarrow \mathbb{R}$ be a primitive function for a function $g\colon J \longrightarrow \mathbb{R}$. Then there exists an indefinite integral of a function $(g \circ f) \cdot f'$ and

$$\int g(f(x))f'(x)\,dx = G(f(x)) + c$$

Proof:
Because f and G are differentiable, then their superposition is also differentiable and

$$(G \circ f)'(x) = G'(f(x))f'(x) = g(f(x))f'(x).$$

Taking an integral of both sides completes the proof.

□

Example 9

$$\int \sin^2 x \cos x\,dx \underset{dt=\cos x\,dx}{\overset{t=\sin x}{=}} \int t^2\,dt = \frac{1}{3}t^3 + c = \frac{1}{3}\sin^3 x + c.$$

Example 10

$$\int e^{x^2} x\,dx \underset{dt=2x\,dx}{\overset{t=x^2}{=}} \frac{1}{2}\int e^t\,dt = \frac{1}{2}e^t + c = \frac{1}{2}e^{x^2} + c.$$

Example 11

$$\int \frac{x^3}{x^4+6}\,dx \underset{dt=4x^3\,dx}{\overset{t=x^4+6}{=}} \frac{1}{4}\int \frac{1}{t}\,dt = \frac{1}{4}\ln|t| + c = \frac{1}{4}\ln|x^4+6| + c.$$

Partial fraction expansions:

Example 12 (to solve at home)

$$\begin{aligned}\int \frac{2x^2+2x+13}{(x-2)(x^2+1)^2}\,dx &= \int \left(\frac{1}{x-2} - \frac{x+2}{x^2+1} - \frac{3x+4}{(x^2+1)^2}\right) dx \\ &= \frac{1}{2}\cdot\frac{3-4x}{x^2+1} + \frac{1}{2}\ln\frac{(x-2)^2}{x^2+1} - 4\arctan x + c.\end{aligned}$$

Definite integral:

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a)$$

Example 13

$$\int_1^2 (3x^2 - 2x + 1)\,dx = \left(x^3 - x^2 + x\right)\Big|_1^2 = \left(2^3 - 2^2 + 2\right) - \left(1^3 - 1^2 + 1\right) = 14 - 1 = 13.$$

Example 14 (to solve at home)
Calculate the area of the shape limited by the curves $y = x$, $y = 2x$, $xy = 1$, $xy = 2$ for $x > 0$ and $y > 0$.